6.4 Exponential Growth and Decay

DEFINITION Separable Differential Equation

A differential equation of the form $dy/dx = f(y)g(x)$ is called **separable.** We **separate the variables** by writing it in the form

$$\frac{1}{f(y)}\,dy = g(x)\,dx.$$

The solution is found by antidifferentiating each side with respect to its thusly isolated variable.

EXAMPLE 1 Solving by Separation of Variables

Solve for y if $dy/dx = (xy)^2$ and $y = 1$ when $x = 1$.

$$\frac{dy}{dx} = (xy)^2$$

$$\frac{dy}{dx} = x^2y^2$$

$$\int \frac{dy}{y^2} = \int x^2\,dx$$

$$\int y^{-2}\,dy$$

$$-y^{-1} = \frac{1}{3}x^3 + C$$

$$-\frac{1}{y} = \frac{1}{3}x^3 + C$$

$$-\frac{1}{1} = \frac{1}{3}(1)^3 + C$$

$$-1 = \frac{1}{3} + C$$

$$C = -\frac{4}{3}$$

$$-\frac{1}{y} = \frac{1}{3}x^3 - \frac{4}{3}$$

$$-\frac{1}{y} = \frac{x^3-4}{3}$$

$$\frac{1}{y} = \frac{-x^3+4}{3}$$

$$y = \frac{3}{-x^3+4}$$

The differential equation that describes this growth is $dy/dt = ky$, where k is the *growth constant* (if positive) or the *decay constant* (if negative). We can solve this equation by separating the variables.

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}\,dy = k\,dt \qquad \text{Separate the variables}$$

$$\ln|y| = kt + C \qquad \text{Antidifferentiate both sides}$$

$$|y| = e^{kt+C} \qquad \text{Exponentiate both sides}$$

$$|y| = e^{C}\,e^{kt} \qquad \text{Property of exponents}$$

$$y = \pm e^{C}\,e^{kt} \qquad \text{Definition of absolute value}$$

$$y = Ae^{kt} \qquad \text{Let } A = \pm e^{C}.$$

This solution shows that the *only* growth function that results in a growth rate proportional to the amount present is, in fact, exponential. Note that the constant A is the amount present when $t = 0$, so it is usually denoted y_0.

The Law of Exponential Change

If y changes at a rate proportional to the amount present (that is, if $dy/dt = ky$), and if $y = y_0$ when $t = 0$, then

$$y = y_0\,e^{kt}.$$

The constant k is the **growth constant** if $k > 0$ or the **decay constant** if $k < 0$.

Continuously Compounded Interest

Suppose that A_0 dollars are invested at a fixed annual interest rate r (expressed as a decimal). If interest is added to the account k times a year, the amount of money present after t years is

$$A(t) = A_0\left(1 + \frac{r}{k}\right)^{kt}.$$

Differential equation: $\dfrac{dA}{dt} = rA$

Initial condition: $A(0) = A_0$

The amount of money in the account after t years is then

$$A(t) = A_0 e^{rt}.$$

Interest paid according to this formula is said to be **compounded continuously.** The number r is the **continuous interest rate.**

EXAMPLE 2 Compounding Interest Continuously

Suppose you deposit \$800 in an account that pays 6.3% annual interest. How much will you have 8 years later if the interest is (**a**) compounded continuously? (**b**) compounded quarterly?

$y_0 = 800 \quad k = .063 \quad t = 8$

a) $y = 800e^{.063(8)}$

$\approx \$1324.26$

b) $y = 800\left(1 + \frac{.063}{4}\right)^{4(8)}$

$\approx \$1319.07$

Radioactivity

When an atom emits some of its mass as radiation, the remainder of the atom reforms to make an atom of some new element. This process of radiation and change is **radioactive decay,**

$$y = y_0 e^{-kt}, \quad k > 0.$$

The **half-life** of a radioactive element is the time required for half of the radioactive nuclei present in a sample to decay.

$y = \frac{1}{2} y_0$

EXAMPLE 3 Finding Half-Life

Find the half-life of a radioactive substance with decay equation $y = y_0 e^{-kt}$ and show that the half-life depends only on k.

$$\frac{1}{2} y_0 = y_0 e^{-kt}$$

$$\frac{1}{2} = e^{-kt}$$

$$\ln \frac{1}{2} = -kt$$

$$t = \frac{-\ln \frac{1}{2}}{k} = \frac{-(\ln 1 - \ln 2)}{k}$$

DEFINITION Half-life

The **half-life** of a radioactive substance with rate constant k ($k > 0$) is

$$\text{half-life} = \frac{\ln 2}{k}.$$

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Modeling Growth with Other Bases

$$y = y_0\, b^{ht},$$

where b is any positive number not equal to 1, and h is another rate constant, related to k by the equation $k = h \ln b$.

EXPLORATION 1 **Choosing a Convenient Base**

A certain population y is growing at a continuous rate so that the population doubles every 5 years.

1. Let $y = y_0\, 2^{ht}$. Since $y = 2\, y_0$ when $t = 5$, what is h? What is the relationship of h to the doubling period?

$2y_0 = y_0\, 2^{h(5)}$ $2 = 2^{h(5)}$ $h = \frac{1}{5}$

2. How long does it take for the population to triple?

≈ 7.97 yrs

A certain population y is growing at a continuous rate so that the population triples every 10 years.

3. Let $y = y_0\, 3^{ht}$. Since $y = 3\, y_0$ when $t = 10$, what is h? What is the relationship of h to the tripling period?

$y = y_0\, 3^{1/10\, t}$

4. How long does it take for the population to double? ≈ 6.31 yrs

A certain isotope of sodium (Na-24) has a half-life of 15 hours. That is, half the atoms of Na-24 disintegrate into another nuclear form in fifteen hours.

5. Let $A = A_0(1/2)^{ht}$. Since $y = (1/2)\, y_0$ when $t = 15$, what is h? What is the relationship of h to the half-life?

$y = y_0\, {1/2}^{1/15\, t}$

6. How long does it take for the amount of radioactive material to decay to 10% of the original amount?

$.1 y_0 = y_0\, {1/2}^{1/15\, t}$

≈ 49.83 yrs

It is important to note that while the exponential growth model $y = y_0\, b^{ht}$ satisfies the differential equation $dy/dt = ky$ for any positive base b, it is only when $b = e$ that the growth constant k appears in the exponent as the coefficient of t. In general, the coefficient of t is the reciprocal of the time period required for the population to grow (or decay) by a factor of b.

EXAMPLE 4 Choosing a Base

At the beginning of the summer, the population of a hive of bald-faced hornets (which are actually wasps) is growing at a rate proportional to the population. From a population of 10 on May 1, the number of hornets grows to 50 in thirty days. If the growth continues to follow the same model, how many days after May 1 will the population reach 100?

EXAMPLE 5 Using Carbon-14 Dating

Scientists who use carbon-14 dating use 5700 years for its half-life. Find the age of a sample in which 10% of the radioactive nuclei originally present have decayed.